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Statistics of resonances in a semi-infinite disordered chain

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We study the average density of resonances (DOR) for a semi-infinite disordered chain, coupled to the outside world by a (semi-infinite) perfect lead. A set of equations is derived, which provides the general framework for calculating the average DOR, for an arbitrary disorder and coupling strength. These general equations are applied to the case of weak coupling and an asymptotically exact expression for the averaged DOR is derived, in the limit of small resonance width. This expression is universal, in the sense that it holds for any degree of disorder and everywhere in the (unperturbed) energy band.

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I. INTRODUCTION

Open quantum systems often exhibit the phenomenon of resonances. Resonances correspond to quasistationary states which have a long lifetime but eventually decay into the continuum. (A particle, initially within the system, escapes to infinity.) One approach to the problem of resonances is based on the study of the analytic properties of the scattering matrix, $S(E)$, in the complex energy plane. Resonances correspond to the poles, $\tilde{E}_\alpha = E_\alpha - \frac{i}{2}\Gamma_\alpha$, of $S(E)$ on the unphysical sheet.[1,2] An alternative approach amounts to solving the stationary Schrödinger equation, with the boundary condition of an outgoing wave only. This condition, which makes the problem non-Hermitian, describes a particle ejected from the system. The Schrödinger equation with such boundary condition admits complex eigenvalues $\tilde{E}_\alpha$, which correspond to the resonances.[1,2] This kind of approach leads in a natural way to a non-Hermitian effective Hamiltonian, and it has been used for a long time in scattering theory, including scattering in chaotic and disordered systems (see Refs. 3 and 4, and references therein). There are many examples of resonances in atomic and nuclear physics. More recently, there has been much interest in resonant phenomena in the field of chaotic and disordered systems (for a recent review, see Ref. 4).

There is considerable amount of work concerning the distribution $P(\Gamma)$ of resonance widths in one-dimensional disordered chains.[5–9] Numerical studies presented in that work demonstrate that, in a broad range of Γ, $P(\Gamma) \propto \Gamma^{-\gamma}$, with the exponent γ being close to 1. (A similar behavior pertains also to two- and three-dimensional systems with localized states[7,9]). An analytical calculation[8] has been performed for a one-dimensional continuous (white-noise) potential, in the semiclassical limit, when the localization length is large. It has been shown in that work that in a broad range of Γ, $P(\Gamma)$ is well fitted by a function $\Gamma^{-1.25}$. For sufficiently small Γ, however, $(1/\Gamma)$ behavior was obtained, followed by a sharp cutoff at still smaller Γ, due to the finite size of the sample. A different analytical approach was recently developed in Ref. 10, for a discrete tight-binding random chain. The limit of strong disorder (i.e., opposite to that of Ref. 8) was considered and the $(1/\Gamma)$ behavior (for a semi-infinite chain) was derived.

In the present paper we develop an approach to the problem of resonances. The approach is based on counting the number of poles of a resolvent of the corresponding non-Hermitian Hamiltonian. We derive a set of equations for a semi-infinite disordered chain, coupled to a (semi-infinite) lead. This set contains, in principle, the full solution of the problem, for an arbitrary coupling strength and disorder. The equations simplify considerably in the weak coupling regime. For this case we rigorously derive the asymptotically exact $(1/\Gamma)$ rule and present a simple scaling formula, which contains only the product of the localization length and the density of states, at the relevant energy.

II. MODEL AND ITS EFFECTIVE HAMILTONIAN

The system is depicted in Fig. 1. Black dots, labeled by $n=1,2,\ldots$, designate sites along the semi-infinite disordered chain. Each site of the chain is assigned a site energy, ϵ_n. Different ϵ_n's $(n=1,2,\ldots)$ are independent random variables chosen from some distribution $q(\epsilon)$. Open circles, labeled by $n=0,-1,-2,\ldots$, represent a perfect semi-infinite lead to which the chain is coupled. All sites of the lead are assigned $\epsilon_n=0$. The lead simulates the free space outside the chain. All nearest neighbor sites of the chain are coupled to each other by a hopping amplitude t, and the same is true for all nearest neighbor sites of the lead. The only exception to this rule is the pair (0,1) which provides coupling between the chain and the lead. The hopping amplitude for this pair is taken to be equal to t'. This allows us to tune the coupling from $t'=0$ (closed chain) to $t'=t$ (fully coupled chain).

As was mentioned in the Introduction, the most direct approach to the problem of resonances is based on solving the stationary Schrödinger equation with the boundary condition of an *outgoing wave* (see Fig. 1). The Schrödinger

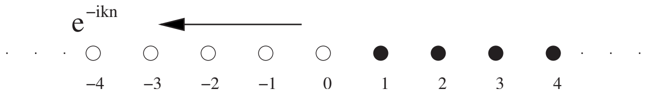

FIG. 1. A one-dimensional chain coupled to a lead. Sites of the disordered chain are denoted by black dots. Sites of the lead are denoted by open circles. The arrow represents the outgoing wave.

equation for the entire system (chain+lead) is a set of coupled equations

$$-t\psi_{n+1}-t\psi_{n-1}=\tilde{E}\psi_n \quad (\text{for } n<0), \tag{1}$$

$$-t\psi_{-1}-t'\psi_1=\tilde{E}\psi_0 \quad (n=0), \tag{2}$$

$$-t\psi_2-t'\psi_0+\epsilon_1\psi_1=\tilde{E}\psi_1 \quad (n=1), \tag{3}$$

$$-t\psi_{n+1}-t\psi_{n-1}+\epsilon_n\psi_n=\tilde{E}\psi_n \quad (n>1). \tag{4}$$

We recall that $\epsilon_n=0$ for $n<1$ (the lead) and it is random for $n\geqslant 1$ (the chain). Equations (1)–(4) are to be solved subjected to the boundary condition of an outgoing wave in the lead, i.e., $\psi_n \propto \exp(-i\tilde{k}n)$, for $n\leqslant 0$, with $\operatorname{Im}\tilde{k}<0$. The complex wave vector $\tilde{k}$ is related to $\tilde{E}$ by $\tilde{E}=-2t\cos\tilde{k}$. Using the plane wave shape of the outgoing wave in the lead, it is straightforward to eliminate from Eqs. (1)–(4) all ψ_n's with $n<1$, thus reducing the problem to a system of equations for the amplitudes ψ_n on the sites of the disordered chain alone $(n=1,2,\ldots)$:

$$-t\psi_{n+1}-t\psi_{n-1}+\tilde{\epsilon}_n\psi_n=\tilde{E}\psi_n \quad (n=1,2,\ldots) \tag{5}$$

with the condition $\psi_0=0$. Here $\tilde{\epsilon}_n=\epsilon_n$ for $n=2,3,\ldots$, but not for $n=1$. This end site is assigned a complex energy

$$\tilde{\epsilon}_1=\epsilon_1-t\eta\exp i\tilde{k}, \tag{6}$$

where the parameter $\eta=(t'/t)^2$ describes the coupling strength to the outside world. Thus, the resonances are given by the complex eigenvalues of the non-Hermitian effective Hamiltonian defined in Eq. (5). This non-Hermitian Hamiltonian, $\tilde{H}$, differs from the Hermitian Hamiltonian, H, of the corresponding closed system (i.e., with $\eta=0$) only by the complex correction to the energy of the first site [Eq. (6)] as follows:

$$\tilde{H}=H-t\eta e^{i\tilde{k}}P, \tag{7}$$

where P is the projection on site 1. We set the energy scale by taking $t=1$ and denote by z the complex variable $\tilde{E}/t$. Note that, since $\tilde{k}$ is related to z via $z=-2\cos\tilde{k}$, the effective Hamiltonian $\tilde{H}$ depends on z. Thus, Eq. (5) does not constitute a standard eigenvalue problem and the eigenvalues of $\tilde{H}$ have to be determined self-consistently. Defining formally the resolvent $\tilde{G}(z)=(z-\tilde{H})^{-1}$ we can write

$$\begin{aligned}\tilde{G}&=(z-H+\eta e^{i\tilde{k}}P)^{-1}\\&=[(z-H)(1+\eta e^{i\tilde{k}}GP)]^{-1}\\&=\frac{1}{1+\eta e^{i\tilde{k}}GP}G,\end{aligned} \tag{8}$$

where $G=(z-H)^{-1}$ is the "unperturbed" resolvent. Resonances correspond to the singularities of the matrix $\tilde{G}_{nm}(z)$, in the complex z plane, or to the roots of the equation

$$1+\eta e^{i\tilde{k}(z)}G_{11}(z)=0, \tag{9}$$

where $G_{11}(z)$ is the (1,1) element of G in the site representation.

III. EXPRESSION FOR THE AVERAGE DENSITY OF RESONANCES

Let us write $G_{11}(z)$ as

$$G_{11}(z)=\frac{1}{z-\epsilon_1-S_1(z)}, \tag{10}$$

where $S_1(z)$ is the self-energy for site 1. Equation (9) can be written as

$$F(z)\equiv z-\epsilon_1-S_1(z)+\eta e^{i\tilde{k}(z)}=0. \tag{11}$$

Denoting the zeros of $F(z)$ by z_α, we write the density of resonances (DOR), for a given realization of the disorder, as $\rho(z)=\Sigma_\alpha\delta(z-z_\alpha)$. This expression refers directly to the semi-infinite chain and it should be understood as the $N\to\infty$ limit of the analogous expression for a finite chain, of size N. The limit is well defined for any $\operatorname{Im} z\neq 0$ and no division of the sum by N is necessary,[10] in contrast to the usual case of the density of states (on the real axis) for a Hermitian problem.

Here we see a crucial difference between the open system ($\eta\neq 0$) and the closed one ($\eta=0$). In the closed system, all N roots of Eq. (11) are real and in order to obtain a well defined $N\to\infty$ limit, we would need to divide $\rho(z)$ by N, getting in the limit the density of states. However, in the open system, the roots of Eq. (11) are complex and the average value of $\rho(z)$ itself has a well defined limit, which is what we call the density of resonances.

It is convenient to define a δ function of a complex variable F as $\delta(F)\equiv\delta(\operatorname{Re}F)\delta(\operatorname{Im}F)$ and to use the relation

$$\sum_\alpha \delta(z-z_\alpha)=|F'(z)|^2\delta(F(z)), \tag{12}$$

where the prime indicates a derivative with respect to z. This relation is a generalization of the corresponding relation for a real variable. It is valid if $F(z)$ is analytic in a domain containing the zeros z_α and $F'(z_\alpha)\neq 0$. Indeed, under such conditions in the neighborhood of a zero, the equation $w=F(z)$ can be solved by an analytic function $z=z(w)$ and the Jacobian of the change of variables from $(\operatorname{Re} w,\operatorname{Im} w)$ to $(\operatorname{Re} z,\operatorname{Im} z)$ is $|F'(z)|^2$ due to the Cauchy-Riemann equations. We will assume that $F'(z_\alpha)\neq 0$ with probability 1. Substituting Eq. (11) for $F(z)$, we have

$$\rho(z)=|1-S_1'(z)+\eta\lambda'(z)|^2\delta(z-\epsilon_1-S_1(z)+\eta\lambda(z)), \tag{13}$$

where $\lambda(z)$ stands for the expression $e^{i\tilde{k}(z)}$.

Next we average Eq. (13) over all realizations of the set $\{\epsilon_n\}$ of the random site energies (we denote this average by angular brackets). Since the self-energy $S_1(z)$ does not depend on the energy ϵ_1 of the first site, we can single out the variable ϵ_1 and average over it explicitly. This leads to the following expression for the average DOR:

$$\langle\rho(z)\rangle = \int d\epsilon q(\epsilon)\langle\{|1+\eta\lambda'|^2 - 2\,\mathrm{Re}[(1+\eta\lambda')S_1'] + |S_1'|^2\}\delta(z-\epsilon-S_1+\eta\lambda)\rangle. \tag{14}$$

The two correlated random variables, S_1 and S_1', depend on the set $\{\epsilon_2,\epsilon_3,\ldots\}$ of the random site energies. Note that we do not need to know the joint probability distribution for S_1 and S_1'. Indeed, only three special combinations of these two variables appear in Eq. (14). It is therefore useful to introduce three functions

$$f(w) = \langle\delta(S_1-w)\rangle, \tag{15}$$

$$g(w) = \langle S_1'(z)\delta(S_1-w)\rangle, \tag{16}$$

$$h(w) = \langle|S_1'(z)|^2\delta(S_1-w)\rangle, \tag{17}$$

and write Eq. (14) as

$$\langle\rho(z)\rangle = \int d\epsilon q(\epsilon)[|1+\eta\lambda'(z)|^2 f(z-\epsilon+\eta\lambda(z)) - 2\,\mathrm{Re}(1+\eta\lambda'(z))g(z-\epsilon+\eta\lambda(z)) + h(z-\epsilon+\eta\lambda(z))]. \tag{18}$$

We emphasize that S_1 refers to a closed (semi-infinite) chain, so that one can use the standard relations for various quantities of the Hermitian problem.[11] In particular, we will need the relation

$$S_1(z) = \frac{1}{z-\epsilon_2-S_2(z)}, \tag{19}$$

where $S_2(z)$ is the self-energy for site 2, with site 1 excluded. It immediately follows that

$$S_1'(z) = -S_1^2(z)(1-S_2'(z)). \tag{20}$$

With the help of the definitions (15)–(17) and the recursion relations (19) and (20) it is straightforward to derive integral relations between the functions f, g, and h. For instance, using Eq. (20), $h(w)$ can be rewritten as

$$h(w) = \left\langle (1-2\,\mathrm{Re}\,S_2' + |S_2'|^2)\,\delta\!\left(z-\epsilon_2-S_2-\frac{1}{w}\right)\right\rangle. \tag{21}$$

The variable ϵ_2 is now "isolated" [i.e., all other quantities in Eq. (21) do not depend on this variable] and can be averaged over. This leads to

$$h(w) = \int d\epsilon q(\epsilon)\left[f\!\left(z-\epsilon-\frac{1}{w}\right) - 2\,\mathrm{Re}\,g\!\left(z-\epsilon-\frac{1}{w}\right) + h\!\left(z-\epsilon-\frac{1}{w}\right)\right]. \tag{22}$$

The two other equations can be derived in a similar way, yielding

$$g(w) = \frac{w^2}{|w|^4}\int d\epsilon q(\epsilon)\left[g\!\left(z-\epsilon-\frac{1}{w}\right) - f\!\left(z-\epsilon-\frac{1}{w}\right)\right] \tag{23}$$

and

$$f(w) = \frac{1}{|w|^4}\int d\epsilon q(\epsilon) f\!\left(z-\epsilon-\frac{1}{w}\right). \tag{24}$$

Let us stress that the above defined functions of the complex argument w are not complex valued functions in the usual sense but rather shorthand notations for a function of two real variables, w_1 and w_2. For instance, Eq. (24) for the function $f(w)$, which is the probability distribution for the real and imaginary parts of the self-energy $S_1=w_1+iw_2$, can be written more explicitly as

$$f(w_1,w_2) = \frac{1}{(w_1^2+w_2^2)^2}\int d\epsilon q(\epsilon) f \times\left(x-\epsilon-\frac{w_1}{w_1^2+w_2^2},\ y+\frac{w_2}{w_1^2+w_2^2}\right). \tag{25}$$

The three equations, Eqs. (22)–(24), supplemented by the expression (18) provide the general framework for computing the average DOR, $\langle\rho(z)\rangle$, in the complex plane $z=x+iy$, for any strength of disorder and for an arbitrary coupling η. We do not attempt to solve the problem in its full generality but rather restrict ourselves to the small η, i.e., weak coupling case.

IV. WEAK COUPLING LIMIT

From now on we assume that the coupling constant η is small and develop a "linear response theory" with respect to η. In this limit the width of all resonances becomes proportional to η, so that the "cloud" of resonances in the (x,y) plane gets squeezed toward the real axes. In order to define a meaningful $\eta\to 0$ limit for the DOR, one must stretch the y axis by a factor of $1/\eta$. This rescaling of the imaginary part of the energy is crucial, and explains why the weak coupling ($\eta\to 0$) limit of the open system will differ significantly from the $\eta=0$ case describing a closed system. It is natural to switch from the original variable $z=x+iy$ to a new variable, $Z=X+iY$, where $X=x$ and $Y=-y/\eta$, where the minus sign accounts for the fact that the resonances are located in the lower half plane of the complex variable z. Thus, in the Z plane resonances appear in the upper half-plane and Y is the resonance width, in units of η. The limiting (i.e., in the $\eta\to 0$ limit) average DOR, $\langle\tilde{\rho}(X,Y)\rangle$, in the complex Z plane, is related to the original DOR, $\langle\rho(x,y)\rangle$, as

$$\langle\tilde{\rho}(X,Y)\rangle = \lim_{\eta\to 0}\eta\langle\rho(X,-\eta Y)\rangle. \tag{26}$$

Next, we turn our attention to the functions f, g, and h in the $\eta\to 0$ limit. The function $f(w_1,w_2)$ is the probability distribution for the real and imaginary parts of the self-energy S_1. The shape of this function depends on the point $z=x+iy$. In the $\eta\to 0$ limit, as has been just explained, one should set $y=-\eta Y$, where Y does not depend on η. Furthermore, since the imaginary part of S_1 is proportional to η, we set $w_2=\eta W_2$. The distribution, $F(W_1,W_2)$, for the new variables $(W_1=w_1, W_2=\eta^{-1}w_2)$ is related to the distribution $f(w_1,w_2)$ for the "old" variables by $F(W_1,W_2)=\eta f(W_1,\eta W_2)$. The distribution $F(W_1,W_2)$ has a well defined $\eta\to 0$ limit and it satisfies an integral equation, obtained from Eq. (25) by transforming to the new variables and taking the $\eta\to 0$ limit as follows:

$$F(W_1,W_2)=\frac{1}{W_1^4}\int d\epsilon q(\epsilon)F\left(X-\epsilon-\frac{1}{W_1},\ -Y+\frac{W_2}{W_1^2}\right). \tag{27}$$

The simplifying feature of the $\eta\to 0$ limit is that functions g and h are easily expressible in terms of f. More precisely, we have to define functions G and H of the new variables as follows:

$$G(W_1,W_2)=\eta g(W_1,\eta W_2),\quad H(W_1,W_2)=\eta h(W_1,\eta W_2). \tag{28}$$

These functions, in complete analogy with $F(W_1,W_2)$, have a well defined $\eta\to 0$ limit and satisfy integral relations which are derived from Eqs. (22) and (23) by writing them in explicit form [compare to Eq. (25)], transforming to the new variables and taking the $\eta\to 0$ limit. This yields

$$G(W_1,W_2)=\frac{1}{W_1^2}\int d\epsilon q(\epsilon)\left[G\left(X-\epsilon-\frac{1}{W_1},\ -Y+\frac{W_2}{W_1^2}\right)-F\left(X-\epsilon-\frac{1}{W_1},\ -Y+\frac{W_2}{W_1^2}\right)\right] \tag{29}$$

and

$$H(W_1,W_2)=\int d\epsilon q(\epsilon)\left[H\left(X-\epsilon-\frac{1}{W_1},\ -Y+\frac{W_2}{W_1^2}\right)-2\,\mathrm{Re}\,G\left(X-\epsilon-\frac{1}{W_1},\ -Y+\frac{W_2}{W_1^2}\right)+F\left(X-\epsilon-\frac{1}{W_1},\ -Y+\frac{W_2}{W_1^2}\right)\right]. \tag{30}$$

One can easily check that Eqs. (29) and (30) are satisfied by

$$G(W_1,W_2)=\frac{W_2}{Y}F(W_1,W_2),\quad H(W_1,W_2)=\left(\frac{W_2}{Y}\right)^2F(W_1,W_2). \tag{31}$$

Now we can derive an expression for $\langle\tilde{\rho}(X,Y)\rangle\langle\tilde{\rho}(X,y)\rangle$, in terms of the function F, by taking the $\eta\to 0$ limit in Eq. (18) and using Eqs. (26) and (31) as follows:

$$\langle\tilde{\rho}(X,Y)\rangle=\left(\frac{\mathrm{Im}\,\lambda(X)}{Y}\right)^2\int d\epsilon q(\epsilon)F(X-\epsilon,-Y+\mathrm{Im}\,\lambda(X)). \tag{32}$$

Note that, since $z=X+i\eta Y$, in the $\eta\to 0$ limit $\tilde{k}(z)$ approaches $k(X)$ which is related to X by $X=-2\cos k$. Therefore $\mathrm{Im}\,\lambda$ in Eq. (32) is given by

$$\mathrm{Im}\,\lambda=\sqrt{1-\frac{X^2}{4}}. \tag{33}$$

It is convenient to define a new variable, $J=W_2/Y$, so that J is the imaginary part of self-energy (the resonance width) in units of $Y\eta$. Finally, we denote W_1 by R (the real part of the self-energy) and define the probability distribution $P(R,J)$, instead of $F(W_1,W_2)$. $P(R,J)$ satisfies the integral equation

$$P(R,J)=\frac{1}{R^4}\int dR'q\left(x-\frac{1}{R}-R'\right)P\left(R',\frac{J}{R^2}-1\right), \tag{34}$$

where the integration variable ϵ was replaced by $R'=x-\epsilon-R^{-1}$. The expression (32) for the average DOR now reads

$$\langle\tilde{\rho}(X,Y)\rangle=\frac{1}{Y^3}\left(1-\frac{X^2}{4}\right)\int dRq(X-R)P\left(R,\ \frac{1}{Y}\sqrt{1-\frac{X^2}{4}}-1\right). \tag{35}$$

While performing integration in Eqs. (34) and (35), one should keep in mind that the function $P(R,J)$ is identically zero for $R>\sqrt{J}$. This property follows from the basic recursion relation for the self-energy S.

The integral equation (34), supplemented by the expression (35), completely defines the problem of resonances in the weak coupling limit. Note that the coupling strength, η, does not appear explicitly in these equations. It only determines the units in which the resonance width is measured (the widths of all resonances are proportional to η). Thus, $\langle\tilde{\rho}(X,Y)\rangle$ is determined solely by the properties of the closed system and, in this sense, Eqs. (34) and (35) describe the regime of linear response with respect to the coupling strength η. The function $P(R,J)$ describes the joint probability distribution for the real and imaginary parts of the self-energy (of the closed system and for z approaches the real axis) and the integral equation (34) has appeared previously in the study of Anderson localization.[13] Integration of Eq. (34) over J yields the integral equation for the probability distribution, $P(R)$, of the real part of the self-energy as follows:

$$P(R)=\frac{1}{R^2}\int dR'q\left(x-\frac{1}{R}-R'\right)P(R'). \tag{36}$$

This equation is very useful in the theory of one-dimensional localization,[11,12] because the knowledge of $P(R)$ allows one to compute the localization length, ξ, according to

$$\frac{1}{\xi(X)} = \int dR' P(R') \ln|R'|. \tag{37}$$

We shall use this relation in the next section.

V. LARGE-*J* ASYMPTOTICS

We were not able to obtain the complete analytic solution of the integral equation (34). It is possible, however, to find the large-J asymptotics of the function $P(R,J)$. We state the result and give the proof later: To leading order in the small quantity, $1/J$,

$$P(R,J) = \frac{\nu(X)\xi(X)}{2J^2}, \tag{38}$$

where $\nu(X)$ is the usual density of states, on the real energy axis, for the closed chain. An important feature of this result is that the large-J asymptotics does not contain R. Therefore, by plugging Eq. (38) into Eq. (35) we immediately obtain the simple result for the small-Y asymptotic of the average DOR as follows:

$$\langle\tilde{\rho}(X,Y)\rangle = \frac{\nu(X)\xi(X)}{2Y}. \tag{39}$$

Note that the factors containing $1-\frac{X^2}{4}\equiv D^2$ had canceled, so that X enters only via the density of states and the localization length. This cancellation is the consequence of the obvious scaling property of the general equation (35), namely, the dependence on D drops out if one transforms to a new variable, $Y\to Y/D$, and to a new function, $\rho\to D\rho$. Thus, the asymptotic $(1/Y)$ behavior is universal, in the sense that it holds for any degree of disorder and for any $-2<X<2$.

It remains to prove the result stated in Eq. (38). It is convenient to define the function

$$W(R,J) = \int_J^\infty dJ' P(R,J') \tag{40}$$

and to study its large-J asymptotics. An integral equation for $W(R,J)$ is derived by integrating Eq. (34) over the second argument of P. While performing this integration one should remember that the function $P(R,J')$ is identically zero for $J'<R^2$. It follows from this property that for $J<R^2$

$$W(R,J) = \int_0^\infty dJ' P(R,J') = P(R) \quad (J<R^2). \tag{41}$$

Combining both cases, i.e., $J<R^2$ and $J>R^2$, one obtains the following integral equation:

$$W(R,J) = \Theta\left(1-\frac{J}{R^2}\right)P(R) + \frac{1}{R^2}\Theta\left(\frac{J}{R^2}-1\right)\int dR' q\left(x-\frac{1}{R} - R'\right)W\left(R',\frac{J}{R^2}-1\right). \tag{42}$$

We are interested in the large-J behavior of $W(R,J)$. It is useful to define the Laplace transform

$$\tilde{W}(R,s) = \int_0^\infty dJ e^{-sJ} W(R,J) \tag{43}$$

and to study its small-s behavior. The integral equation for the transform is obtained directly from Eq. (42) as follows:

$$\tilde{W}(R,s) = \tilde{W}_0(R,s) + e^{-sR^2}\int dR' q\left(x-\frac{1}{R}-R'\right)\tilde{W}(R',sR^2), \tag{44}$$

where

$$\tilde{W}_0(R,s) = \frac{1}{s}(1-e^{-sR^2})P(R). \tag{45}$$

Anticipating that $W(R,J)$ is proportional to $1/J$, we look for the solution (in the small-s limit) of Eq. (44) in the form

$$\tilde{W}(R,s) = A(R)\ln s + B(R) + \cdots. \tag{46}$$

This ansatz satisfies Eq. (44) if

$$A(R) = \int dR' q\left(x-\frac{1}{R}-R'\right)A(R') \tag{47}$$

and

$$B(R) = B_0(R) + \int dR' q\left(x-\frac{1}{R}-R'\right)B(R'), \tag{48}$$

with

$$B_0(R) = R^2 P(R) + A(R)\ln R^2. \tag{49}$$

Since the site energy distribution, $q(\epsilon)$, is normalized to 1, it immediately follows from Eq. (47) that $A(R)$ is a constant. We denote this constant by $-a$ and determine it as follows.

Define a function $\Phi(R')$ which is the nontrivial solution of the integral equation

$$\Phi(R') = \int dR \Phi(R) q\left(x-\frac{1}{R}-R'\right). \tag{50}$$

Note that the kernel $q\left(x-\frac{1}{R}-R'\right)$ is not symmetric with respect to R and R' and that Eq. (50), unlike Eq. (47), is not solved by a constant. The solution of Eq. (50) is

$$\Phi(R') = \frac{1}{R'^2}P\left(\frac{1}{R'}\right). \tag{51}$$

We now go back to Eq. (48), multiply it by $\Phi(R)$, integrate over R and, with the help of Eq. (50), obtain

$$\int dR \Phi(R) B_0(R) = 0, \tag{52}$$

which, using Eq. (49) and the aforementioned result $A(R)=\text{const}=-a$, yields the value of a as follows:

$$a = \frac{\int dRP(R)P\left(\frac{1}{R}\right)}{\int dR\frac{1}{R^2}P\left(\frac{1}{R}\right)\ln R^2}, \qquad (53)$$

where the explicit form of $\Phi(R)$ [see Eq. (51)] has been used at the last step. The integral in the numerator is known in localization theory[12] and is equal to the density of states $\nu(X)$. The integral in the denominator is equal to $(2/\xi(X))$ [see Eq. (37)]. Thus, in the small-s limit, $\widetilde{W}(R,s) = -a \ln s$, with $a = \nu(X)\xi(X)/2$. As a consequence, in the large-J limit, $W(R,J) = a/J$ and $P(R,J) = a/J^2$ which completes the proof of Eq. (38). One can also show that Eq. (48) has a unique solution $B(R)$ and the behavior of $\widetilde{W}(R,s)$ suggests that $W(R,J) = \alpha/J + R^2B(R)/J^2 + \cdots$ and $p(R,J) = \alpha/J^2 + 2B(R)R^2/J^3 + \cdots$.

VI. CONCLUSIONS

We studied the average density of resonances (DOR) for a semi-infinite disordered chain, coupled to the outside world by a (semi-infinite) perfect lead. The main result of this work is the expression (18) for the average DOR, supplemented by the three integral equations, Eqs. (22)–(24), for the three functions, h, g, and f. This set of equations provides the general framework for calculating the average DOR, for an arbitrary disorder and coupling strength. We applied these general equations to the case of weak coupling and derived an asymptotically exact expression for the average DOR, in the limit of small resonance width [Eq. (39)]. This expression is universal, in the sense that it holds for any degree of disorder and everywhere in the (unperturbed) energy band.

It is worthwhile to emphasize the essential difference between the average DOR, as defined in this paper, and the probability distribution of the resonance widths, often used in the literature (see, e.g., Ref. 5). For a finite size chain, of N sites, the two quantities differ only by a factor N. However, when N approaches ∞, the probability distribution shrinks toward a delta function, while the average DOR approaches a perfectly well defined limit. Since we have taken a semi-infinite chain from the start, it was essential for us to work with the average DOR, rather than with the probability distribution of the resonant widths. This enabled us to obtain the $1/Y$ behavior in the $N \to \infty$ limit, in contrast to the statement of Ref. 5 that in that limit the $1/Y$ behavior has no region of applicability.

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